The Fundamental Group

Tyrone Cutler

July 9, 2020

Contents

1 Where do Homotopy Groups Come From?

Working in the based category Top_*, a 'point' of a space X is a map $S^0 \to X$. Unfortunately, the set $Top_*(S^0, X)$ of points of X determines no topological information about the space. The same is true in the homotopy category. The set of 'points' of X in this case is the set

$$\pi_0 X = [S^0, X] = [*, X]_0 \tag{1.1}$$

of its **path components**. As expected, this pointed set is a very coarse invariant of the pointed homotopy type of X. How might we squeeze out some more useful information from it?

One approach is to back up a step and return to the set $Top_*(S^0, X)$ before quotienting out the homotopy relation. As we saw in the first lecture, there is extra information in this set in the form of track homotopies which is discarded upon passage to $[S^0, X]$. Recall our slogan: it matters not only that a map is null homotopic, but also the manner in which it becomes so.

So, taking a cue from algebraic geometry, let us try to understand the automorphism group of the zero map $S^0 \to * \to X$ with regards to this extra structure. If we vary the basepoint of X across all its points, maybe it could be possible to detect information not visible on the level of π_0. Conceivably we might be able to find a way to iterate this idea, and the hope would be that at some stage this information would be sufficient to completely determine the homotopy type of X.

In fact this idea is not so far fetched. At least in that it turns out to be correct for a large class of 'nice' spaces. The nice spaces we have in mind are exactly the CW complexes[1] which were introduced into topology by J.H.C. Whitehead. These spaces are very important and central to our theory. They will be the topic of a subsequent lecture.

Now, returning to our initial thought, what do we mean by an 'automorphism' of the zero map $* : S^0 \to X$. Well, this is exactly a homotopy $F : * \simeq *$. i.e. a homotopy filling in the square

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\overset{F}{\Leftarrow}"] \\ * \arrow[r] & X. \end{tikzcd} \tag{1.2}$$

But in the last lecture we saw how to turn such a square into concrete topological data. The homotopy F describes two paths going from the basepoint of X to the point $F(0, 1/2) \in X$, and these paths glue together to define a map α_F as that induced out of the pushout in the next diagram

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & I \arrow[d] \arrow[ddr, bend left] & \\ I \arrow[r] \arrow[drr, bend right] & S^1 \arrow[dr, dashed, "\alpha_F"] \arrow[ul, phantom, "\lrcorner", very near start] & \\ & & X. \end{tikzcd} \tag{1.3}$$

Thus in search of deeper homotopical information than is contained in $\pi_0 X$ we are naturally led to study the set

$$\pi_1 X = [S^1, X]. \tag{1.4}$$

A moments reflection at this point makes it clear than the argument is repeatable: an automorphism of the zero map $* : S^1 \to X$ describes a map $S^2 \to X$, and so on. In this way we are led to the sets

$$\pi_n X = [S^n, X]. \tag{1.5}$$

So the next question is, if these sets $\pi_n X$ are interesting, then are they computable? What tools are available to aid in their computation? The first useful bit of structure is the fact that $\pi_n X$ has a group structure for $n \geq 1$, and an abelian group structure for $n \geq 2$. Moreover, the structure is functorial, in that a map $f : X \to Y$ induces a homomorphism

$$f_* : \pi_n X \to \pi_n Y. \tag{1.6}$$

As it turns out, for $n \geq 2$, the group structure on $\pi_n X$ that we have in mind is uniquely determined by the requirement that it enjoys this type of functoriality. In any case we can define functors

$$Top_* \xrightarrow{\pi_1} Gro, \qquad Top_* \xrightarrow{\pi_n} Ab, \quad n \geq 2 \tag{1.7}$$

which turn out to be homotopy functors (cf. exercise sheet 1). Thus, if we prefer, we can instead work directly with the homotopy category and consider them to be functors

$$hTop_* \xrightarrow{\pi_1} Gro, \qquad hTop_* \xrightarrow{\pi_n} Ab, \quad n \geq 2. \tag{1.8}$$

[1] Or better yet, the class of all those spaces homotopy equivalent to a CW complex.

2 The Fundamental Group

In this section we fix a pointed space X. Our goal is to turn $\pi_1 X$ into a group. When we have established this structure we will refer to $\pi_1 X$ as the *fundamental group* of X. As we will see, the fact that $\pi_1 X$ is a group is really a consequence of the fact that S^1 is a *cogroup* object in $hTop_*$. This is something which will be formalised at a later point, so in this section let us avoid too much abstraction and consider it as motivation for what will come.

The basic idea is to start with the unit interval $I = [0,1]$, and notice that it has the rather strange property of being freely homeomorphic to its own wedge $I \vee I$, where we identify 0 in one copy of I with 1 in the other. Essentially this says that the operation of composition of paths is tautologically continuous. Next we take S^1 as the quotient $I/\partial I$ and let c be the map induced by the following diagram

$$\begin{tikzcd} I \arrow[r, "\cong"] \arrow[d] & I \vee I \arrow[d] \\ S^1 \arrow[r, dashed, "c"] & S^1 \vee S^1. \end{tikzcd} \tag{2.1}$$

To be specific, we'll take c to be the map

$$c(t) = \begin{cases} (2t, *) & 0 \leq t \leq \frac{1}{2} \\ (*, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}, \qquad t \in T/\partial I. \tag{2.2}$$

If we prefer, we may instead identify S^1 with the unit circle in $\mathbb{C}$, in which case (2.2) becomes the map

$$c(z) = \begin{cases} (z^2, *) & im(z) \geq 0 \\ (*, z^2) & im(z) \leq 0 \end{cases}, \qquad z \in S^1 \subseteq \mathbb{C}. \tag{2.3}$$

In either case, geometrically c is the map which pinches the equatorial S^0 to a point to create the bouquet $S^1 \vee S^1$.

We use c as follows. Given maps $f, g : S^1 \to X$ we let $f + g$ be the map

$$f + g : S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X \tag{2.4}$$

where ∇ is the fold map, defined to be the identity on each summand. From the work in the last lecture (i.e. Le. 1.1 and Co. 1.2) we see that the homotopy class of $f + g$ depends only on the homotopy classes of f and g. Thus we can set

$$[f] + [g] = [f + g] \tag{2.5}$$

to get a well-defined pairing

$$+ : \pi_1 X \times \pi_1 X \to \pi_1 X, \qquad (\alpha, \beta) \mapsto \alpha + \beta. \tag{2.6}$$

This is to be the product operation in $\pi_1 X$. For it to be a group operation it must be unital and associative, and have inverses. Note that despite the additive notation, it will turn out that this product is *not* commutative in general.

We first construct the unit for (2.6). For $i = 1, 2$ let $q_i : S^1 \vee S^1 \to S^1$ be the pinch map, which is the identity on the i^{th} summand and collapses the opposite summand to a point.

Lemma 2.1 *The following diagrams commute up to homotopy*

$$\begin{tikzcd} S^1 \arrow[r, "c"] \arrow[dr, equal] & S^1 \vee S^1 \arrow[d, "q_1"] \\ & S^1 \end{tikzcd} \qquad \begin{tikzcd} S^1 \vee S^1 \arrow[d, "q_2"'] & S^1 \arrow[l, "c"'] \arrow[dl, equal] \\ S^1. & \end{tikzcd} \tag{2.7}$$

Proof We only write down a homotopy $F_s : q_1 c \simeq id_{S^1}$, the other case being similar. With equation (2.2) in mind we have

$$q_1 c(t) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{2} \\ * & \frac{1}{2} \leq t \leq 1 \end{cases}, \qquad t \in I/\partial I. \tag{2.8}$$

Putting

$$F_s(t) = \begin{cases} \frac{2}{1+s} t & 1 \leq t \leq \frac{1+s}{2} \\ * & \frac{1+s}{2} \leq t \leq 1 \end{cases}, \qquad t \in I/\partial I, s \in I \tag{2.9}$$

we get the desired homotopy. ∎

Corollary 2.2 *The constant map* $* : S^1 \to X$ *is a two-sided unit for the product* (2.1)*.*

Proof Let $\alpha \in \pi_1 X$ and choose a representative $f : S^1 \to X$ for it. We show that $f + * \simeq f$, a homotopy $* + f \simeq f$ being constructed similarly. Thus consider the diagram

$$\begin{tikzcd} S^1 \arrow[r, "c"] \arrow[dr, equal] & S^1 \vee S^1 \arrow[r, "f \vee *"] \arrow[d, "q_1"'] & S^1 \vee S^1 \arrow[d, "\nabla"] \\ & S^1 \arrow[r, "f"] & S^1. \end{tikzcd} \tag{2.10}$$

Notice that the clockwise composite around the diagram is $f + *$ while the composition in the opposite direction is f. Now, the square in the diagram commutes strictly and the triangle commutes by Lemma 2.1. In particular the whole diagram homotopy commutes and gives us the homotopy $f + * \simeq f$. In terms of the original class α, this is the equation $\alpha + * = \alpha$. ∎

Next we address associativity.

Lemma 2.3 *The diagram*

$$\begin{tikzcd} S^1 \arrow[r, "c"] \arrow[d, "c"'] & S^1 \vee S^1 \arrow[d, "c \vee 1"] \\ S^1 \vee S^1 \arrow[r, "1 \vee c"] & S^1 \vee S^1 \vee S^1 \end{tikzcd} \tag{2.11}$$

commutes up to homotopy.

Proof We have

$$(c \vee 1)c(t) = \begin{cases} (4t, *, *) & 0 \leq t \leq \frac{1}{4} \\ (*, 4t-1, *) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ (*, *, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad (1 \vee c)c(t) = \begin{cases} (2t, *, *) & 0 \leq t \leq \frac{1}{2} \\ (*, 4t-2, *) & \frac{1}{2} \leq t \leq \frac{3}{4} \\ (*, *, 4t-3) & \frac{3}{4} \leq t \leq 1. \end{cases} \tag{2.12}$$

To complete the proof it will be sufficient to show that both these maps are homotopic to

$$t \mapsto \begin{cases} (3t, *, *) & 0 \le t \le \frac{1}{3} \\ (*, 3t-1, *) & \frac{1}{3} \le t \le \frac{2}{3} \\ (*, *, 3t-2) & \frac{2}{3} \le t \le 1. \end{cases} \tag{2.13}$$

We'll write down the homotopy for $(c \vee 1)c$, and leave the task of writing down the other homotopy to the reader, since it is similar.

The homotopy we need is not difficult construct. Basically, it involves mapping each subinterval linearly onto $[0,1]$ and solving an equation of the form $t \mapsto at + b$. The result is the map $G_s : S^1 \times I \to S^1 \vee S^1 \vee S^1$ given by

$$G_s(t) = \begin{cases} ((4-s)t, *, *) & 0 \le t \le \frac{1}{4-s} \\ (*, \frac{(4-s)(2+s)t-(2+s)}{(4-s)(1+s)-(2+s)}, *) & \frac{1}{4-s} \le t \le \frac{1+s}{2+s} \\ (*, *, (2+s)t-(1+s)) & \frac{1+s}{2+s} \le t \le 1 \end{cases}, \qquad t \in I/\partial I, s \in I. \quad \blacksquare \tag{2.14}$$

Corollary 2.4 *The product* (2.1) *is associative. That is, if* $\alpha, \beta, \gamma \in \pi_1 X$*, then*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma). \tag{2.15}$$

Proof In this proof we will begin the standard conceit of denoting with the same symbol, both a homotopy class and a chosen representative for it. As we'll see, this won't lead to confusion. The more correct equation $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$, written in terms of homotopy classes, will then be interpreted in terms of representative maps as $(\alpha+\beta)+\gamma \simeq \alpha+(\beta+\gamma)$.

So, for the proof we notice that $(\alpha + \beta) + \gamma$ and $\alpha + (\beta + \gamma)$ are given by the two ways around the following diagram

$$\begin{tikzcd} & S^1 \vee S^1 \arrow[dr, "c\vee 1"] & & & \\ S^1 \arrow[ur, "c"] \arrow[dr, "c"'] & & S^1 \vee S^1 \vee S^1 \arrow[r, "\alpha\vee\beta\vee\gamma"] & S^1 \vee S^1 \vee S^1 \arrow[r, "\nabla"] & S^1 \\ & S^1 \vee S^1. \arrow[ur, "1\vee c"'] & & & \end{tikzcd} \tag{2.16}$$

The diagram homotopy commutes by Lemma 2.3, and gives us the proof. $\blacksquare$

So, at this stage we have shown that $\pi_1 X$ is a unital monoid. The inverse is the last thing we need to address, and this is supplied by the map

$$\iota : \begin{array}{l} S^1 \to S^1 \\ t \mapsto (1-t). \end{array} \tag{2.17}$$

Lemma 2.5 *The diagrams*

$$\begin{tikzcd} S^1 \arrow[r, "c"] \arrow[d, "*"'] & S^1 \vee S^1 \arrow[d, "1\vee\iota"] \\ S^1 & S^1 \vee S^1 \arrow[l, "\nabla"'] \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r, "c"] \arrow[d, "*"'] & S^1 \vee S^1 \arrow[d, "\iota\vee 1"] \\ S^1 & S^1 \vee S^1 \arrow[l, "\nabla"'] \end{tikzcd} \tag{2.18}$$

both commute up to homotopy.

Proof We prove the statement only for the left-hand diagram. The clockwise composite around this square is the map

$$t \mapsto \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ 2-2t & \frac{1}{2} \le t \le 1 \end{cases} \tag{2.19}$$

and we get the required null homotopy by defining

$$H_s : t \mapsto \begin{cases} 2t & 0 \le t \le \frac{(1-s)}{2} \\ 1-s & \frac{(1-s)}{2} \le t \le \frac{(1+s)}{2} \\ 2-2t & \frac{(1+s)}{2} \le t \le 1. \end{cases} \qquad \blacksquare \tag{2.20}$$

The map ι is implemented as follows. Given $f : S^1 \to X$, we define $-f : S^1 \to X$ to be the composite

$$-f : S^1 \xrightarrow{\iota} S^1 \xrightarrow{f} X. \tag{2.21}$$

Again this passes to homotopy classes and gives a well-defined operation on the fundamental group

$$- : \pi_1 X \to \pi_1 X, \qquad \alpha \mapsto -\alpha = \alpha\iota. \tag{2.22}$$

Corollary 2.6 *The monoid $\pi_1 X$ has inverses. That is, if $\alpha \in \pi_1 X$, then*

$$\alpha + (-\alpha) = * = (-\alpha) + \alpha. \tag{2.23}$$

Proof Consider the diagram

$$\begin{tikzcd} S^1 \vee S^1 \arrow[r, "1\vee\iota"] & S^1 \vee S^1 \arrow[r, "\alpha\vee\alpha"] \arrow[d, "\nabla"] & S^1 \vee S^1 \arrow[d, "\nabla"] \\ S^1 \arrow[u, "c"] \arrow[r, "*"] & S^1 \arrow[r, "\alpha"] & S^1. \end{tikzcd} \tag{2.24}$$

Its clockwise composite is the map $\alpha + (-\alpha)$. The right-hand square commutes strictly, and the left-hand square homotopy commutes by Lemma 2.5. Thus the whole diagram homotopy commutes, and displays a null homotopy $\alpha + (-\alpha) \simeq *$. The null homotopy $(-\alpha) + \alpha \simeq *$ is constructed similarly. $\blacksquare$

Having established the corollary we will now prefer to write simply

$$\alpha - \beta = \alpha + (-\beta), \qquad \alpha, \beta \in \pi_1 X. \tag{2.25}$$

At this stage we have assembled all the pieces.

Proposition 2.7 *If X is a pointed space, then the set $\pi_1 X = [S^1, X]$ has a canonical group structure.*

Proof The product, denoted $+$, is defined by equation (2.6). Corollary 2.2 shows that it is unital, and that the unit is supplied by the class of the constant map $* : S^1 \to X$. Corollary 2.4 shows that $+$ is associative, so the equation

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \tag{2.26}$$

holds for all classes $\alpha, \beta, \gamma \in \pi_1 X$ and can be understood without the need for brackets. Finally, Corollary 2.6 shows that $+$ has inverses, with $-\alpha = \alpha\iota : S^1 \to X$ being the inverse of $\alpha : S^1 \to X$, where ι is defined in (2.21). ∎

Definition 1 *For a pointed space X, the group $\pi_1 X = [S^1, X]$ with the structure constructed in Proposition 2.7 is said to be the* ***fundamental group*** *of X.* □

Later we shall define abelian groups $\pi_n X$ for all $n \geq 2$. For this reason we also call $\pi_1 X$ the first **homotopy group** of X. We stress that $\pi_1 X$ need not be abelian. Something else we would like to stress is that we have attached to $\pi_1 X$ a *particular* choice of group structure. It seems feasible that the underlying set could admit many distinct group structures. It will turn out that the requirement of this structure to be functorial constrains it somewhat. However it still does not render it unique. Essentially this comes down to our choice of the map (2.2), and we will understand in future the consequences of this choice.

Now, it is clear that as a set, $\pi_1 X$ is an invariant of the homotopy type of X, since a pointed map $f : X \to Y$ induces a function $[S^1, X] \to [S^1, Y]$, $[g] \mapsto f_*[g] = [fg]$, which depends only on the homotopy class of f. In fact we can improve this observation and check that $\pi_1 X$ is really an *algebraic* invariant of the homotopy type of X. This is exactly what we need for π_1 to be a group-valued functor.

Proposition 2.8 *A pointed map $f : X \to Y$ induces a group homomorphism*

$$f_* = \pi_1(f) : \pi_1 X \to \pi_1 Y \tag{2.27}$$

which depends only on the pointed homotopy class of f.

Proof It is clear that if $f \simeq g$, then $f_* = g_*$, since this is true on the level of sets. If $\alpha, \beta \in \pi_1 X$, then the diagram

$$\begin{tikzcd} S^1 \arrow[r, "c"] \arrow[d, equal] & S^1 \vee S^1 \arrow[r, "\alpha \vee \beta"] \arrow[d, equal] & S^1 \vee S^1 \arrow[r, "\nabla"] \arrow[d, "f \vee f"] & S^1 \arrow[d, "f"] \\ S^1 \arrow[r, "c"] & S^1 \vee S^1 \arrow[r, "(f\alpha) \vee (f\beta)"] & Y \vee Y \arrow[r, "\nabla"] & Y \end{tikzcd} \tag{2.28}$$

bears witness to the equation

$$f_*(\alpha + \beta) = (f_*\alpha) + (f_*\beta) \tag{2.29}$$

where the left-hand sum is formed in $\pi_1 X$ and the right-hand sum is form in $\pi_1 Y$. This is exactly what we needed to show. ∎

Corollary 2.9 *A pointed homotopy equivalence $X \simeq Y$ induces a group isomorphism $\pi_1 X \cong \pi_1 Y$.* ∎

One last thing which needs to be addressed before closing this section is the issue of basepoints. We have decided to formulate the definition and construction of the fundamental group in terms of pointed spaces and maps. Clearly this has been necessary for everything we have done. The homotopy group $\pi_1 X$ is in fact an invariant of the *pointed* homotopy type of X. As a consequence, there is no way to directly use the functor π_1 to study a given unbased space Y. The only option is to first turn Y into a based space by choosing for it a basepoint. Since there is no canonical way to do this, it completely destroys the functorality of the construction.

Moreover, even if we choose a basepoint $y_0 \in Y$, then the only way a free map $f : Y \to Y'$ can induce a homomorphism is if we give Y' the basepoint $f(y_0)$. In this case we do get a map

$$f_* : \pi_1(Y, y_0) \to \pi_1(Y', f(y_0)) \tag{2.30}$$

but it no longer depends on even the free homotopy class of f. Rather we have instead to work with a fixed representative map, and then turn this free map into a pointed homotopy class.

On the other hand, it can be quite useful to study even pointed spaces through this idea of varying basepoints. For example, if Y_0 is the path-component of Y containing y_0, then $\pi_1(Y, y_0) = \pi_1(Y_0, y_0)$. Thus if we are not prepared to let basepoints vary, then the fundamental group will only ever see information coming from the basepoint component.

There is other evidence to support this too. It turns out that the group $\pi_1(Y_0, y_0)$ is - up to isomorphism - independent of the particular choice of basepoint y_0 from within its path class Y_0. Thus, in a way, we can still make sense of the fundamental group of a unbased space as long as it is path-connected. Of course, while doing so we lose much information. We replace the group $\pi_1(Y_0, y_0)$ with its isomorphism class. In this way we encounter the same kind of loss of information we saw in the topological category when passing from maps and homotopies to homotopy classes.

We'll revisit the dependence on basepoints later and make it precise. The statement we would like to make is really a consequence of something deeper which relates the sets of pointed and unpointed homotopy classes between any two given pointed spaces. In any case, the discussion above means that the following definition has meaning in both the pointed and unpointed categories.

Definition 2 *A path-connected space X is said to* ***simply-connected*** *if* $\pi_1 X = 1$. $\square$

3 Methods of Computation

In this section we will fix a pointed space X and discuss some approaches to the computation of its fundamental group. We will be sketchy with details since in many case we will want to surpass these basic results with more intricate statements.

3.1 Covering Spaces

A *covering space* of X is a map $p : E \to X$ such that

1) For each $x \in X$, the fibre $p^{-1}(x)$ is a discrete subspace of E

2) Each point $x \in X$ has a neighbourhood $U \subseteq X$ for which there exists a homeomorphism $U \times p^{-1}(x) \cong X|_U = p^{-1}(U)$ making the following diagram commute strictly

$$\begin{tikzcd} U \times p^{-1}(x) \arrow[rr, "\cong"] \arrow[dr, "pr_U"'] & & X|_U \arrow[dl, "p|"] \\ & U. & \end{tikzcd} \tag{3.1}$$

The full theory of covering spaces is developed in Hatcher [6] § 1.3, tom Dieck [11] § 3, Spanier [10] § 2, and in the smooth category, for instance, by Lee [7] pg. 91, pg. 548. Our discussion of the topic is included only for motivation and will be very limited.

For simplicity we will restrict to the case that X is path connected. Furthermore we will assume that X is locally path connected and semilocally simply connected[2]. Any connected manifold or CW complex satisfies these properties. We call a covering space $p : E \to X$ *connected* if E is connected. We write $F = p^{-1}(*)$ for the *typical fibre* and let $i : F \hookrightarrow E$ be the inclusion. Two covering spaces of X are said to be *isomorphic* if they are homeomorphic over X.

Theorem 3.1 *Assume that X is path connected, locally path connected and semilocally simply connected. Then if $p : E \to X$ is a connected covering, there is a transitive right action of $\pi_1 X$ on F. The induced map $p_* : \pi_1 E \to \pi_1 X$ is injective, and there is a bijection of right $\pi_1 X$-sets*

$$p_*(\pi_1 E)\backslash \pi_1 X \cong F. \quad \blacksquare \tag{3.2}$$

The point is that covering space theory open up access to $\pi_1 X$ through geometric and algebraic techniques. This is especially true in light of the next theorem, which reduces the study of covering spaces over suitable X to the study of just one particular example.

Theorem 3.2 *Assume that X is path connected, locally path connected and semilocally simply connected. Then there exists a connected covering space $p : E \to X$ characterised uniquely up to isomorphism by the property that $\pi_1 E = 1$. The group $\pi_1 X$ acts freely on E from the left by covering isomorphisms and X is homeomorphic to the quotient by this action. In particular p is a quotient map. Any other connected covering of X is isomorphic to a projection of the form $H\backslash E \to X$, where $H \subseteq \pi_1 X$ is a subgroup.* $\blacksquare$

A covering space satisfying E the properties of the last theorem is said to be *universal.* Often the universal covering space of X can be identified or explicitly constructed using geometric methods.

Example 3.1 The circle S^1 meets the requirements of Theorem 3.2. Thus it has a universal cover, and it be shown that it is exactly the exponential function $p : \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$. The fibre of p is the integers, and there is an isomorphism of groups $\pi_1 S^1 \cong \mathbb{Z}$. $\square$

[2] A space X is said to be **semilocally simply connected** if each $x \in X$ has a neighbourhood $U \subseteq X$ for which the induced map $\pi_1 U \to \pi_1 X$ is trivial.

Example 3.2 A *graph* is a one-dimensional CW complex. In this context its 0-cells are called *vertices*, and its 1-cells are called *edges*. If X is a connected graph and $p : \widetilde{X} \to X$ is a covering, then $\widetilde{X}$ is also a connected graph. If X is a finite complex, and $p^{-1}(*)$ is a finite set of cardinality n, then $\widetilde{X}$ is finite, and moreover the relation $\chi(\widetilde{X}) = n \cdot \chi(X)$ holds. Here we are quoting May [8] § 4.

Examples of connected, finite graphs include bouquets of circles $\bigvee^n S^1$. The covering spaces of these examples may be constructed explicitly. See Hatcher [6] pg. 58 for pictorial descriptions of some covering spaces of $S^1 \vee S^1$, and a recipe for how to construct others, including its universal cover.

Example 3.3 In this example we consider *Riemann surfaces*, by which we shall here understand to be 2-dimensional connected, orientable, manifolds without boundary. The idea is to study the implications of the *Uniformisation Theorem* [1] for surfaces, which implies that any simply connected Riemann surface is conformally equivalent to one of either *i*) the complex plane $\mathbb{C}$, or *ii*) the upper half plane $H^+ = \{z \in \mathbb{C} \mid im(z) > 0\}$, or *iii*) the Riemann sphere $S^2 \cong \mathbb{C}_\infty$. In particular this implies that any compact surface which is not S^2, cannot be simply connected.

Now, the universal covering of an arbitrary surface must be simply connected by Th. 3.2, and so must be either $\mathbb{C}$, H^+ of S^2 by the Uniformisation Theorem. Thus by another application of 3.2 we can shift the problem of computing the fundamental groups of compact surfaces to the question of which discrete groups admit suitable actions on the three simply connected examples. For example it is know that the only discrete groups which act suitably on H^+ are torsion free, and this implies that the compact surfaces which have H^+ as a universal cover all have torsion free fundamental groups.

In another direction, the classification theorem for surfaces [4] says that any compact surface is homeomorphic to either S^2, or to a connected sum of $g \geq 1$ copies of the torus $T^2 = S^1 \times S^1$. From this point of view the Uniformisation Theorem implies that a compact surface which is not S^2 must have a contractible universal cover. This has interesting applications for the *higher homotopy groups*. □

3.2 The Seifert-van Kampen Theorem

The Seifert-van Kampen Theorem is a tool useful in the computation of the fundmental groups of certain pushouts. In particular it applies to calculate the fundamental group of an adjunction space formed when attaching a cell of dimension ≥ 2 to a connected space. We give the statement next, and for its proof refer the reader to Brown [2] for a conceptual treatment, and to May [8] for a particularly concise account.

Theorem 3.3 (Seifert-van Kampen) *Let X be the union of two path connected subspaces $U, V \subseteq X$. Assume that the interiors $\mathring{U}, \mathring{V}$ cover X and that $U \cap V$ is a path connected subspace which contains the basepoint $* \in X$. Then*

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1 U \arrow[d] \\ \pi_1 V \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \pi_1 X \end{tikzcd} \tag{3.3}$$

is a pushout in the category of groups. ∎

Example 3.4 One of the most useful applications of the Seifert-van Kampen Theorem is to the computation of the fundamental groups of wedge sums. Compared to homotopy classes of maps into a product, homotopy classes of maps into a coproduct are notoriously hard to understand, and it is for this reason that the Seifert-van Kampen Theorem really shines.

As an example, we consider $S^1 \vee S^1$, which is the pushout in Top of the left-hand diagram below. It is not difficult to find suitable neighbourhoods of $*$ which cover $S^1 \vee S^1$ so as to be able to apply the Seifert-van kampen Theorem. Taking for granted the fact that $\pi_1 S^1 \cong \mathbb{Z}$ the theorem gives us the pushout in Gro on the right below

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S^1 \arrow[d] \\ S^1 \arrow[r] & S^1 \vee S^1 \end{tikzcd} \qquad\qquad \begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \pi_1(S^1 \vee S^1). \end{tikzcd} \tag{3.4}$$

This is exatly the statement that

$$\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \tag{3.5}$$

is the free group on two generators. □

References

[1] L. Ahlfors, *Conformal Invariants*, Second Edition, AMS Chelsea, (2010).

[2] R. Brown, *Topology and Groupoids*, Revised Edition, (2006). Available at http://www.groupoids.org.uk/orderbook.html.

[3] D. Ferrario, R. Piccinini, *Simplicial Structure in Topology*, Springer, (2011).

[4] J. Gallier, D. Xu, *A Guide to the Classification Theorem for Compact Surfaces*, Springer, (2013).

[5] R. Geoghegan, *Topological Methods in Group Theory*, Springer, (2008).

[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.

[7] J. Lee, *Introduction to Smooth Manifolds*, Second Edition, Springer, (2003).

[8] P. May, *A Concise Course in Algebraic Topology*, Univeristy of Chicago Press, (1999).

[9] J. Munkres, *Topology*, Second Edition, Prentice-Hall (2000).

[10] E. Spanier,

[10] Algebraic Topology, McGraw-Hill, (1966).

[11] T. tom Dieck, *Algebraic Topology*, European Mathematical Society, (2008).

[12] G. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, New York, (1978).